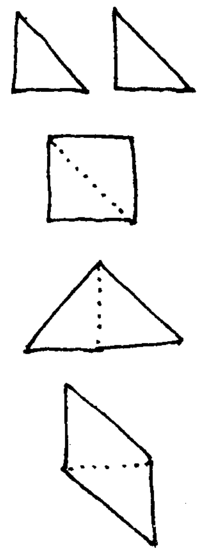
The Tangram Puzzle (page 83)

Extension 1: Area and Perimeter

This activity offers another way to look at the mathematical ideas in *Squaring Up, Double the Circumference*, and *The Area Stays the Same* from the measurement section. The Tangram pieces allow you to think about how area and perimeter relate by comparing shapes instead of resorting to standard units of measurement.

To compare the areas of the square, parallelogram, and middle-size triangle, rearrange the two small triangles to make each of the shapes. The areas of these three shapes are the same. Then match their edges to compare their perimeters. The parallelogram and middle-size triangle have the same-length perimeters, but the perimeter of the square is shorter. This is because the same amount of area is squashed together more compactly in a square, requiring less perimeter. If you rearranged the same area into a long skinny rectangle, you'd need a longer "fence" to surround it and the perimeter would therefore be longer. If you made a circle of the same area, however, the circle's circumference would be shorter than even the square's perimeter; the circle is the most economical shape for minimizing the distance around.



Extension 3: Using All Seven

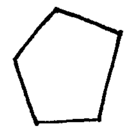
This activity asks you to use all seven pieces of the Tangram to make different convex shapes. A convex shape is one with all of its interior angles measuring less than 180 degrees. Shapes with at least one interior angle greater than 180 degrees are called concave.

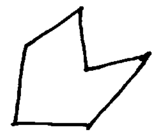
It's easy to spot them — children have told me, "They go in and out," or "They have dents."

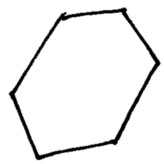
The shapes you build with the Tangram pieces are all polygons. A polygon is a closed shape (which means it encloses an interior region the way a fence encloses a yard) made of connecting straight line segments. Sometimes a polygon is defined as a "simple closed curve." This seems contradictory, since the sides of a polygon are all straight line segments, but the definition has been in common use for quite some time. The "simple" part of the definition is important. It means that the line segments enclose one contiguous interior region; the inside of a polygon can't be divided into smaller regions. One way to think about this is to think of the corners where line segments meet — the vertices — as fence posts. Only two segments of the fence can touch each fence post.

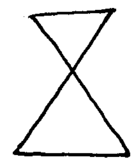
The directions for this extension could have asked you to use the seven Tangram pieces to investigate convex polygons, instead of convex shapes. It's hard to decide when to use standard mathematical terminology or more common language. A good pedagogical approach is to use both — common language to provide access to an idea and correct terminology to help learners extend their mathematical vocabulary.

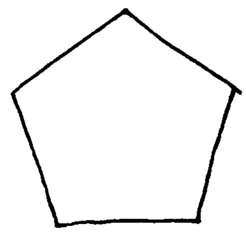
One more comment about the convex polygons you search for in this activity. Polygons are categorized by the number of sides they have — triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, and so on. Any shape with four sides — squares, parallelograms, rectangles, and so on — is a quadrilateral. When a polygon has all identical sides and angles, it's called a regular polygon. A square is a regular polygon; so is an equilateral triangle. When we see pictures of pentagons, hexagons, and octagons, we typically see pictures of them as regular polygons.

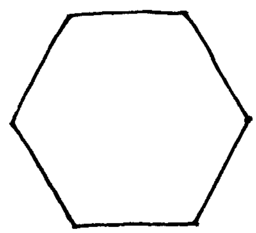

convex

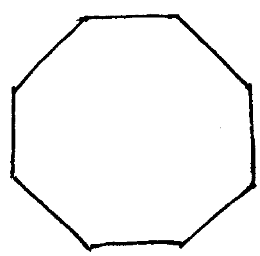

concave


polygon

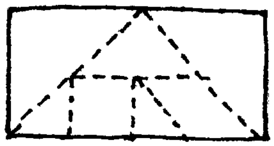

not a polygon


pentagon


hexagon


octagon

The convex shapes you make in this activity, except for the square, are not regular. The rectangle shown below, for example, was made from the seven Tangram pieces, but it isn't a regular quadrilateral because its sides aren't all the same length:



When doing this activity, be sure to name the different polygons you make.

Extension 4: Making Squares

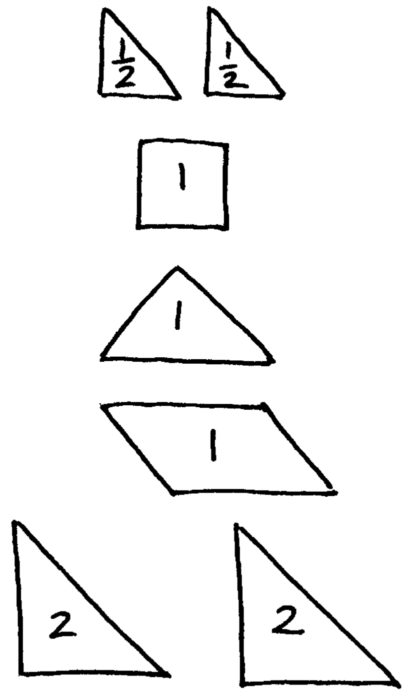
I know it isn't possible to form a square from six Tangram pieces. I can't remember where I heard this, but my own experience experimenting with the Tangram pieces led me to believe that it is true. However, for a long time I wasn't able to explain why. (It's not enough to assert that I tried really hard to make a square from six pieces and couldn't — a proof calls for a convincing argument.) I was stymied. I don't mean to imply that I spent a lot of waking hours thinking about why I couldn't arrange six Tangram pieces into a square. But I remained intrigued by this fact and wondered about it whenever I taught a lesson using the Tangram puzzle.

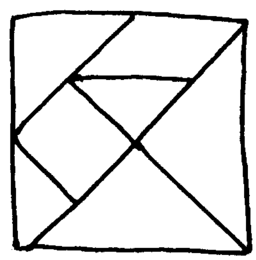
In 1990, I raised this question at the annual retreat I hold with Math Solutions instructors. I formed a panel of four people who had a good deal of mathematical expertise, experience, and confidence but who hadn't yet thought about this particular problem and asked them to think about it together, out loud, while the rest of us observed. My idea was that it would be interesting not only to learn about why a square can't be formed from six Tangram pieces but also to have the chance to see mathematical thinkers in action.

So we observed four people rummage for ways to approach the problem. They cut Tangram pieces; they moved pieces about; they exchanged ideas; at times, one person would retreat into private thoughts and then reemerge to share discoveries; others would build on these ideas, as interested in how the others thought as they were in their own ideas. The group functioned in the way I want groups of students in the classroom to function.

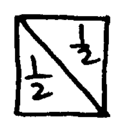
Because of this experience I finally learned why making a six-piece square isn't possible. I offer my understanding knowing that it's hard to follow someone else's reasoning but hoping that my explanation will be useful to you and, perhaps, get you interested in this problem if you haven't been up to now.

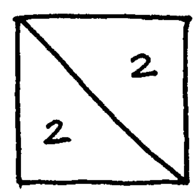
Think about the small square Tangram piece as having an area of one square unit. As you may have learned from the first extension activity, the parallelogram and middle-sized triangle have the same area — also one square unit. The two small triangles that together form the square, parallelogram, and middle-size triangle each have an area of one-half square unit. Since the large triangles can be made from a square and two small triangles, they each have an area of two square units. Using these values, the square formed by all seven Tangram pieces has an area of eight square units. (Check to make sure you are with me this far; this information will be important a bit later.)



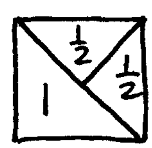


Now think about making a square with 2, 3, 4, 5, 6, and 7 Tangram pieces. First, you can make a square with two pieces by using the two small triangles. This square is the same size as the small square Tangram piece and therefore has an area of one square unit. And because you can find the area of a square by multiplying the length of a side by itself, it measures one unit on a side. If you use the two large triangles to make a square, however, each side measures two units (it's equal in length to two sides of a square), and its area is four square units. (Use your Tangram pieces to check that this is so.) This also makes sense because $2 \times 2 = 4$.



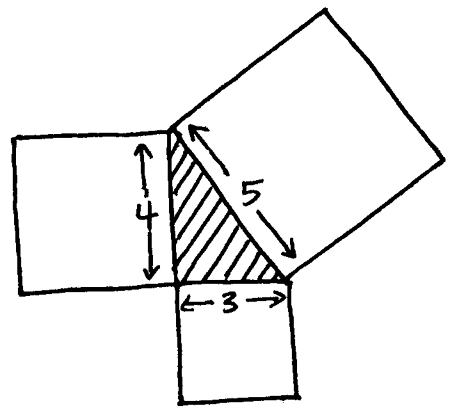


What about using three pieces to make a square? The chart on page 83 shows how to use two small triangles and the middle-size triangle to make a square. Its area is two square units. Each of its sides is equal in length to the long side of the small triangle — its hypotenuse. This is more than one unit but less than two units. How much is that, exactly? Although two is a small number that's generally easy for computations, in this situation it's easier to figure the length of a side for squares with larger but more cooperative areas. For a square with an area of 49 square units, for example, the sides are seven units (since $7 \times 7 = 49$). For a square with an area of 64 square units, the sides are eight units. In these examples, 49 and 64 are square numbers. (Take 49 or 64 pennies and you could arrange either number into a



square array.) And 7 and 8 are the square roots of 49 and 64. We could say that the sides of those two squares are, respectively, $\sqrt{49}$ and the $\sqrt{64}$. Finding the square root of 2 isn't so friendly, so we could use a calculator, pressing 2 and then the square root key (my calculator displays 1.4142135, which makes sense because it's in between 1 and 2). Or we could just say that the side is $\sqrt{2}$ long.

Another way to figure the length of this side is to recall the Pythagorean theorem, which states that for a right triangle (and all of the Tangram triangles are right triangles because they all have a right angle), the hypotenuse (the long side) is equal to the square root of the sum of the squares of the other two sides. The theorem is often written as $a^2 + b^2 = c^2$. The illustration of a right triangle with sides that measure 3 and 4 and a hypotenuse that measures 5, and showing the sides squared, may help you see how the Pythagorean theorem makes sense. And the theorem can be a helpful clue for understanding this problem. To use the Pythagorean theorem to figure the length of the hypotenuse of the small Tangram triangle, I use the information that the shorter sides are each one unit, the same as the length of the side of the small square. The hypotenuse is the square root of $1^2 + 1^2$, which is $\sqrt{2}$. Then I use my calculator as above to get that messy number of 1.4142135.



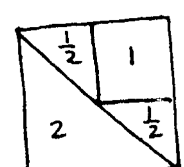
$(3 \times 3) + (4 \times 4) = (5 \times 5)$

$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

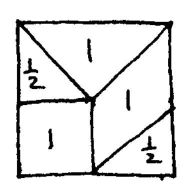
Another clue is that if you examine the sides of all the Tangram pieces, they either are one unit, twice that (two units), $\sqrt{2}$ units, or twice that ($2\sqrt{2}$ units). (Again, use your Tangram pieces to make sense of this information.)

On to making a square with four pieces. Ah, that's possible. First I make a two-piece square using the two large triangles and then I replace one of the triangles with three pieces — the two small triangles and the square. The resulting four-piece square still has an area of four square units and measures two units on a side.



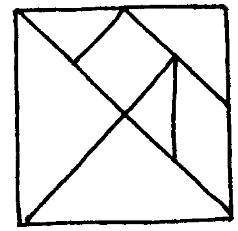
four pieces

A five-piece square can be made from the five small pieces. It's the same size as the four-piece square — four square units — and its sides measure two units.



five pieces

I already know about the seven-piece square. That's the square made from all seven Tangram pieces. Its area is eight square units and each of its sides is the length of the long side (the hypotenuse) of the large Tangram triangle piece. This is the same as twice the hypotenuse of the small triangle, which measures $\sqrt{2}$. So each side of the seven-piece square measures $\sqrt{2} + \sqrt{2}$, which can also be written as $2\sqrt{2}$.



seven pieces

For a square to be formed from six pieces, one piece of the Tangram puzzle has to be eliminated. The choices are to eliminate either one of the small triangles, one-half square unit; the square, parallelogram, or middle-sized triangle, one square unit; or a large triangle, two square units. So the area of a six-piece square would measure either seven-and-a-half square units, seven square units, or six square units. The side of a six-piece square, therefore, would have to measure either 17 1/2, 17, or 16. But there aren't any Tangram pieces with sides any of those lengths or combinations of those lengths. All we have are sides of lengths 1, 2, $\sqrt{2}$, or $2\sqrt{2}$, so it isn't possible to form a square with six Tangram pieces.

Convinced? If not, give it time. Remember that I thought about this problem for a long time — years — before finally making sense of it for myself. Try again when you're interested and rested. And don't forget the Tangram pieces; they contain many of the clues to thinking about this problem.

That's Just Half the Story *(page 84)*

This is the first of several activities that use the uppercase alphabet letters for geometric explorations. *That's Just Half the Story* investigates which of the letters have mirror symmetry, also called line symmetry. Some teachers make cutout block letters available, and children fold them to test for symmetry instead of using mirrors. This works well.

The letters that work only one way have exactly one line of symmetry — A, B, C, D, E, K, M, T, U, V, W, and Y.

Some letters have more than one line of symmetry — H, I, O, and X.

And some letters have zero lines of symmetry — F, G, J, L, N, P, Q, R, S, Z.

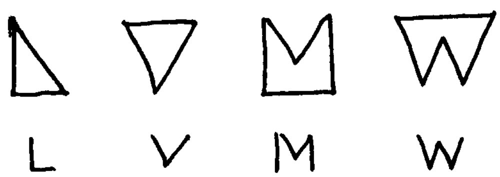
It's also possible for a shape to have an infinite number of lines of symmetry. If the O is formed as a perfect circle, for example, as long as a fold goes through the center of the circle, the two halves will match. Any line that goes through the center of a circle is a line of symmetry.

When children do this activity, they're often surprised and unsettled to find that N, S, and Z do not have lines of symmetry. I think this is because the letters appear to have a kind of balance that letters like F and G, for example, don't have. While N, S, and Z don't have mirror symmetry, they have rotational symmetry. This means that if you imagine each as a cutout letter attached to a surface with a pin through its center, you can rotate the letter 180 degrees so that it's upside down and it will look the same. Notice that the letters that have more than one line of symmetry also have rotational symmetry.

A note about rotational symmetry: Shapes can have rotational symmetry at other than 180 degrees. A square, for example, has rotational symmetry at 90 degrees, and an equilateral triangle has rotational symmetry at 120 degrees.

Interior Regions (page 84)

It's generally easy for children to sort the letters into those that do and don't have interior regions, so there's not much to discuss about this activity. But in the spirit of looking for connections among activities, revisiting this activity after writing the discussion for extension 3, Using All Seven, from *The Tangram Puzzle* made me wonder whether any uppercase letters were polygons. (If you aren't sure what a polygon is, read the explanation for Using All Seven.) The only letter formed only by straight line segments that enclose an interior region is A, but it has extra "tails," a feature that eliminates it from the polygon classification. Some of the letters that don't have interior regions would be polygons if their two dangling line segments were joined. For example, L and V would become triangles; M and W would become pentagons, actually concave pentagons. (Again, read the Using All Seven explanation if you're not sure what makes a shape concave.)

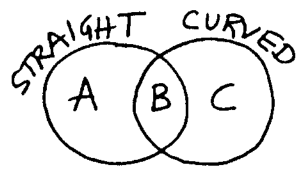


Straight or Curved (page 84)

Deciding which of the uppercase letters have only straight line segments, curves, or both is another easy sorting activity. Converting the chart to a Venn diagram, however, may be something new. A Venn diagram is a clever way to display sets of information when there is information that belongs in more than one set. (Venn diagrams are named for John Venn, a British mathematician, 1834–1923, who worked in statistics, probability, and logic.)

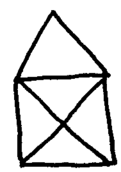
If this activity is your class's first experience with a Venn diagram, it may help to use cutout letters and two circles of different-color yarn. First form the two circles so that they don't overlap, then designate one for letters with straight line segments and one for letters with curves. The problem of where to put letters that have both straight line segments and curves comes up fairly quickly. Ask the children for suggestions. Some may suggest a third yarn enclosure, but that doesn't solve the problem of a letter such as B belonging in all three sets. Some students may suggest making three Bs; you can counter that you only have one of each letter and push them to think of an easier way than cutting out more.

If no one suggests overlapping the circles, then tell them about Venn's idea. Put the B in the intersection and have them verify that it's in both the yarn circle for letters with straight line segments and the yarn circle for letters with curves. While ideas like making three Bs might work, Venn diagrams have become accepted convention for displaying information like this, so children should become familiar with how they work.



No-Lift Letters and More No-Lift Letters (page 85)

The house shape shown in *No-Lift Letters* is one that fascinated me as a young child. Once I realized that I could draw the house without lifting my pencil or retracing a line if I drew it in a certain way, I practiced and practiced until I could do it correctly every time. Then I experimented with other shapes.



Much later, from doing an activity like *More No-Lift Letters*, experimenting convinced me that the only shapes that could be drawn this way were those with zero or two odd vertices. Now I had a way to predict before I tried whether it was possible to draw a shape without lifting the pencil or retracing any line.

It was even later that I began to understand why it was possible to draw shapes with zero or two odd vertices without lifting the pencil or retracing any line. It seemed so strange that this was a rule that worked. Then I engaged in a different kind of experimenting, trying to understand why shapes with 1, 3, or more odd vertices didn't work. I came to see that an odd vertex was a dead end to be avoided, so zero odd vertices was best. Two odd vertices are okay because they provide places to begin and end. More than two odd vertices are trouble. And I haven't yet figured out how to draw a shape with just one odd vertex; maybe it's not possible.

My personal passage through this investigation shows how the same mathematical experiences can be appreciated in different ways at different ages. I returned to this problem over and over again for years. The discovery about odd and even vertices wouldn't have interested me as a young child trying to master how to draw a shape without lifting my pencil or retracing any line. Later, however, learning the rule was exciting to me, like uncovering a secret. But it wasn't until even later that I became interested in why the rule made sense. And perhaps there's still more to think about with this investigation — it wouldn't surprise me.

I sometimes hear teachers worry that students have done an activity in a previous year and therefore will no longer be interested. That hasn't been the case for me. I've found that children often enjoy returning to things they've previously learned. I've also found that revisits offer children the opportunity to see a situation in new ways and to bring to it their new maturity, experience, and learning.

Square Partitioning (page 89)

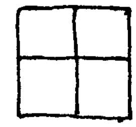
This activity demonstrates how rummaging around and getting immersed in a problem can reveal useful patterns that aren't obvious at first. At least that's what happened to me. For example, from fiddling with this activity I discovered a tried-and-true system for partitioning squares into any number of smaller squares in this sequence — 4, 7, 10, 13, 16, 19, and so on — with the numbers continuing to increase by 3. When you divide a square into four equal-size squares, the change from one square to four squares represents an increase of three squares, so the system is to divide any existing square into four equal-size squares. All of the numbers in the sequence are one more than a multiple of 3.

The next one that seemed easy to do was to partition a square into nine smaller squares, making a three-by-three array. Once I had nine squares, I could divide any square using the system I had discovered before and get three more squares. So this took care of 9, 12, 15, 18, 21, and so on. Or I could divide any of the nine squares into nine more squares, which added eight more squares, to solve the problem for 17, 25, 33, 41, 49, and so on.

But I was getting ahead of the problem with these large number solutions. What about partitioning a square into six, eight, or eleven squares? I stumbled into six from erasing some lines from the nine-square. And I found eight and eleven from dividing a square into a five-by-five array and then erasing some interior lines.

But enough. The fun in a problem like this is the discovering — if you like that sort of thing. I do. Once I have systems, I'm no longer interested, although at times I become delighted all over again when someone gives me a completely new way to look at a problem I thought I completely understood. The lesson is — be mathematically curious. It's a key to becoming mathematically competent.

So what's the story with 5? It's not possible to partition a square into five smaller squares, a reminder that not every mathematical problem has a solution.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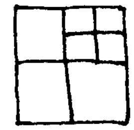
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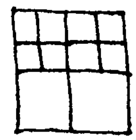
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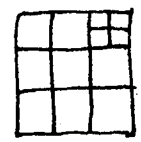
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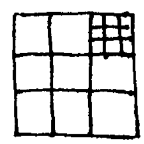
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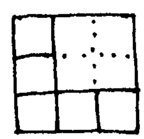
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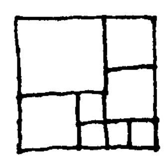
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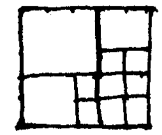
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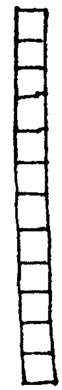
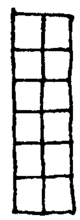
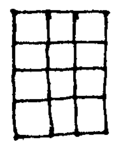
The Banquet Table Problem (page 89)

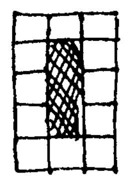
This problem uses the context of arranging tables for a banquet to explore the ideas of area and perimeter. Different arrangements of 12 square tables all have the same area but have different perimeters and, therefore, can seat different numbers of people. For example, putting all 12 tables in one row to make a long, thin table seats the most people — 12 on each side and one at each end for a total of 26. Arranging the tables into a 6-by-2 rectangle results in seating for 16 people — six on each side and two at each end. A 4-by-3 arrangement seats 14 people — four on each side and three on each end. These are the only rectangular arrangements that I had in mind when I wrote the problem. However, students have found ways to arrange the tables in other ways, leaving holes in the center.

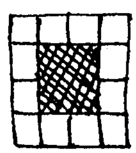






From time to time, students interpret a problem in a different way from the one intended. It's difficult to give precise directions all the time. My first impulse when a student built a table with a "hole" was to clarify that this wasn't what I meant, that I had meant only "filled-in" rectangles. I was locked into the goal I had for the lesson, for students to see the relationship between the number of small tables and the pairs of factors of that number. (For 12, the pairs of factors are 1×12, 2×6, and 3×4.) I wanted them to see that long, thin tables had longer perimeters than did more squarelike tables. When a student suggested a banquet table with a hole, and then others argued about whether or not people could sit inside the hole, I felt that I was losing mathematical control over the lesson.





I relaxed, however, and decided to see the diversion not as a potential disaster but as a way for students to follow their curiosity. The students continued to search for all possible tables and record their findings. Later, in a class discussion about the banquet tables made from 12, 24, and 100 small tables, I asked them to consider just those tables that were "filled-in" rectangles. This focused us on looking at the mathematics I felt was important.

For additional activities that also address area, perimeter, and relationships between them, see *Squaring Up*, on page 54, and *The Area Stays the Same*, on page 57. Also, *Spaghetti and Meatballs for All!*, a children's book I wrote, uses the context of a family reunion to show how rearranging tables can affect seating.

The Four-Triangle Problem (page 93)

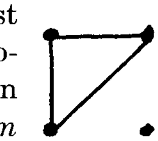
This problem is an extremely versatile investigation that is valuable for and accessible to children at all grade levels. It helps students explore geometric concepts, learn geometric vocabulary in context, and develop spatial reasoning skills.

The triangles used for the investigation are right triangles; they all have one square corner. This is because the triangles come from cutting squares on the diagonal, leaving one of the square's right angles intact in each triangle. Also, all of the triangles are the same size, making them congruent right triangles. It's possible to arrange four congruent right triangles into a larger triangle and also into five different quadrilaterals, two different pentagons, and six different hexagons.

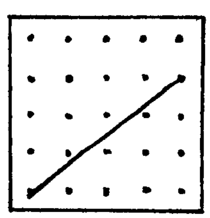
If you're interested in reading more about how to use this activity in the classroom, chapter 9 (pages 99–106) in *A Collection of Math Lessons From Grades 1 Through 3* describes the lesson in a first-grade class. And the *Math By All Means* unit *Geometry, Grades 3–4* (pages 16–25 and 77–82), offers a detailed explanation of the activity with older children.

Geoboard Line Segments (page 96)

There are 14 different-length line segments you can make on the geoboard by stretching rubber bands so that pegs of the geoboard are the endpoints of the segments. Four of them are parallel to the sides of the geoboard, measuring 1, 2, 3, and 4 units. Then there are four that are on the diagonal from one side or corner to the opposite side or corner. Children often think that the diagonal line segments are also 1, 2, 3, and 4 units, but that's not right. The line segment from a peg to the next diagonal peg is longer than the line segment from a peg to the peg directly above or below it. You can use a ruler or string to test that this is so. Or you can think about the Pythagorean theorem, since the diagonal is the hypotenuse of the right triangle. (You can read about the Pythagorean theorem in the explanation about extension 4, Making Squares, of *The Tangram Puzzle*.)



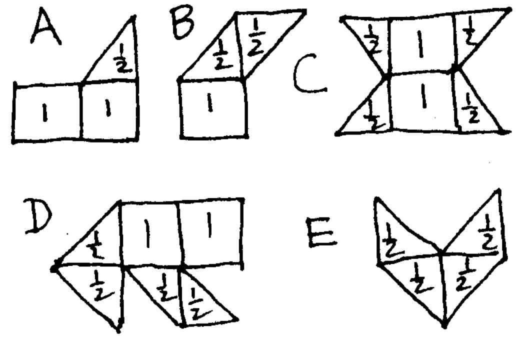
In addition to these eight line segments, there are six more, not parallel to any of the eight I described so far. The line segment that goes from one of the bottom corner pegs to the opposite top corner peg is the longest on the geoboard. The next to the longest goes from one bottom corner peg to the peg just under the opposite top corner peg. I think this is correct because the diagonal line segments are all hypotenuses of right triangles, and this particular one is the hypotenuse of a right triangle whose sides measure 4 units and 3 units. Use the Pythagorean theorem to calculate that this hypotenuse is 5 units, longer than the longest segment that is parallel to the sides of the geoboard. No other right triangle, except for one with both sides 4 units, has sides this large.



Area on the Geoboard (page 99)

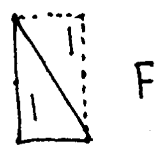
If you can figure the area of the 26 shapes on page 99, you can probably figure the area of any shape on a geoboard. To do so, there are a few techniques that are useful.

One is to get good at spotting halves of squares. In shape A, for instance, there are two whole squares and half of another; its area is 2 1/2 square units. In shape B, there is one square intact topped by a parallelogram that is formed from two half squares. Keep an eye out for those halves and you'll do fine with shapes A through E. You can also confirm these areas by using small paper squares the size of square units, cutting them in half, and fitting them together to make all of the shapes from A to E.

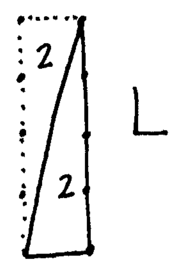


The next shapes get tricky. You can continue to use small paper squares, cutting them to fit the odd-looking shapes and adding up the bits and pieces. One problem with this approach is that measurement is never exact, and the lack of precision will be frustrating. Trust me, it's better to use your head than paper and scissors, at least for these problems.

Here's my advice. Think about rectangles. They're the easiest to figure because you only have to count whole squares. Look at shape F. It's not a rectangle, I realize, so I make it into one. The rectangle I drew has two square units; F is half of it; therefore, F is worth 1 square unit. (Okay, if you'd like, verify this by the paper-and-scissors method.)

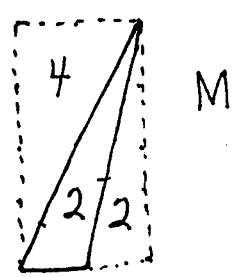


Look at L. If I "rectangulate" it, the rectangle will have an area of 4 square units, so the area of L is 2 square units.



Nice work, you say, but you took the easy ones. What about M? Or N? Or O or P?

My advice is: Relax and rectangulate. My additional advice is that it's always easier to figure out what's outside the shape but inside your rectangle. Look at what happens when I draw a rectangle to enclose M. Its area is 8 square units. The triangle on the right side is the same as L and, therefore, has an area of 2 square units. The triangle on the left is half of the rectangle, so its area is 4 square units. Subtract these two areas from the whole rectangle, and you're left with 2 square units for M.



If you do N, O, and P the same way, you'll find that they each have an area of 2 square units. Does it seem peculiar that L, M, N, O, and P all have the same area? It made sense to me once I noticed that they all have the same-length base and the same height (which is the perpendicular distance from the base to the opposite vertex). Also, I know that you can figure the area of a triangle by multiplying the base times the height and dividing by 2 ($A = bh/2$). Looking at the illustration of "rectangulating" L helps show this; the area of the rectangle can be found by multiplying its base times its height ($A = bh$), and the triangle is half as big. Because L, M , N, O, and P all have the same base and height, their areas are also all the same.

With these techniques — rectangulate, figure out what's outside the shape but inside the rectangle, then subtract — I can figure the area of all the shapes on the page. Below are the answers (all in square units) so that you can check the answers you get.

A = 2 1/2
B = 2
C = 4
D = 4
E = 2
F = 1
G = 2
H = 1
I = 3 1/2
J = 3 1/2
K = 5 1/2
L, M, N, O, P = 2
Q = 3 1/2
R = 3
S = 2 1/2
T = 6
U = 6 1/2
V = 4 1/2
W = 8
X = 4
Y = 9
Z = 8 1/2